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BENDING OF BARS UNDER A FOLLOWER LOAD

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Exact solutions of the problem of nonlinear bending of a thin bar under a point follower load are given. The problem is studied for an arbitrary follower angle and the particular cases of axial and transverse follower forces are considered. The solutions are written in unified parametric form and expressed in terms of Jacobi elliptic functions.

Key words: *bending of bars, follower load, Jacobi elliptic functions.*

Introduction. The behavior and stability of flexible bars under the action of follower loads have been studied extensively (see, e.g., [1, 2]). However, stability analysis of bars under this loading has been based on the geometrically linear equations of equilibrium and motion of the bars. For this purpose, as a rule, the dynamic stability criterion has been used.

Zakharov et al. [3, 4] developed a geometrically nonlinear theory for bending of thin bars using, the static stability criterion of bars based on nonlinear pendulum-type equations. The bar bending shapes were obtained in analytic form and classified for various dead loads and clamping conditions. The solutions were expressed in terms of elliptic integrals and Jacobi functions dependent on a single parameter, namely, the elliptic-function modulus, determined by the boundary conditions and external force, unlike in [5, 6], where the solutions depend on three parameters. Solutions similar to those of [3, 4] can be found in [7], where the secondary loss of stability of the bar was studied.

In the present paper, an exact analytical solution to the problem of nonlinear bending of a thin elastic bar loaded by a point follower force with a specified follower angle is constructed. One end of the bar is rigidly clamped and the other end is free. The critical loads are calculated and equilibrium configurations of the bent bar are found.

General Solution of the Bar Bending Problem. We consider a thin inextensible bar of length L and flexural rigidity EI. The Cartesian coordinate system xOy is chosen in such a manner that the Ox axis is directed along the undeformed straight bar and the coordinate origin is located at its left end. The left end of the bar is clamped and the right end is clamped or free. The bar is compressed by a point force P applied to the right end at a certain angle to the Ox axis. We denote the bar length by l, the angle between the tangent to the bar axis and the Ox axis by $\theta(l)$, and the Cartesian components of the force P by P_x and P_y. The coordinate system is shown in Fig. 1.

We give the general solution of this problem, following [4], where the angle between the direction of the force and the Ox axis is kept constant and denoted by φ_0. In the present paper, the angle between the direction of the follower force and the Ox axis (denoted by φ) can vary during loading.

The equilibrium equations for the bar have the form

$$EI\,\frac{d^2\theta}{dl^2} - P_x \sin\theta + P_y \cos\theta = 0. \tag{1}$$

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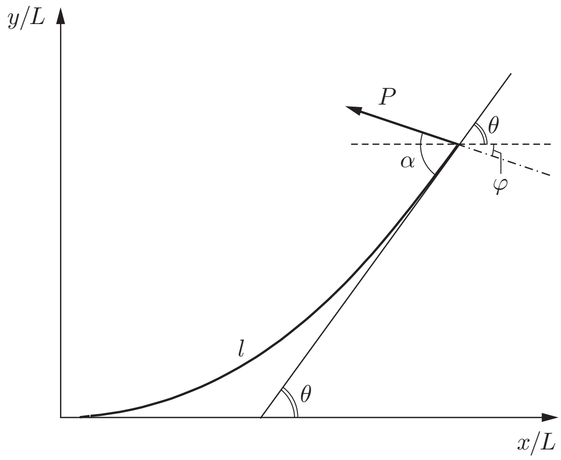

Fig. 1. Coordinate system.

We introduce the following notation: P is the magnitude of the point force, $t = l/L$ is the nondimensional length which varies from 0 to 1, and $q^2 = PL^2/EI$ is an eigenvalue. The solution of Eq. (1) is given by

$$\theta(t) = -\varphi + 2\arcsin\,[k\,\mathrm{sn}\,(qt + F_1, k)], \qquad \frac{d\theta(t)}{dt} = 2kq\,\mathrm{cn}\,(qt + F_1, k), \tag{2}$$

where sn and cn are the Jacobi elliptic sine and cosine. The elliptic-function modulus k and the parameter F_1 are the constants of integration, whose relation to the force P and the angle φ is defined by the boundary conditions of the bar in each particular case.

We introduce the notation for the argument of the elliptic functions

$$u = qt + F_1. \tag{3}$$

Integrating the relations $dx/dl = \cos\theta$ and $dy/dl = \sin\theta$ with allowance for (3), we obtain the coordinates of an arbitrary point of the bar

$$x/L = X_0\cos\varphi + Y_0\sin\varphi, \qquad y/L = Y_0\cos\varphi - X_0\sin\varphi, \tag{4}$$

where

$$X_0 = -t + 2[E(\mathrm{am}\,u) - E(\mathrm{am}\,F_1)]/q, \qquad Y_0 = 2k[\mathrm{cn}\,F_1 - \mathrm{cn}\,u]/q. \tag{5}$$

Here $E(\mathrm{am}\,u)$ is an incomplete elliptic integral of the second kind of the Jacobi elliptic amplitude. Expressions (4) and (5) define the bar bending shapes in parametric form, where the reduced length t plays the role of the parameter.

Bending of a Cantilever Bar under a Follower Force with an Arbitrary Follower Angle. For a cantilever bar, the boundary conditions have the form

$$\theta(0) = 0, \qquad \frac{d\theta(L)}{dl} = 0. \tag{6}$$

We introduce the follower angle α, i.e., the angle between the direction of the force and the tangent to the bar deflection line at the bar end (see Fig. 1). According to the definition of a follower force, we obtain the additional constancy condition for the angle α at the bar end

$$\theta(1) = \alpha - \varphi. \tag{7}$$

We assume that the quantities P and α are the parameters of state determined by the conditions of the problem.

Taking into account the first condition in (6) and using solution (2) of Eq. (1), we obtain $\mathrm{sn}\,F_1 = \sin(\varphi/2)/k$ and, hence,

$$F_1 = F[\arcsin\,(\sin\,(\varphi/2)/k), k], \tag{8}$$

where $F(\varphi, k)$ is an incomplete elliptic integral of the first kind.

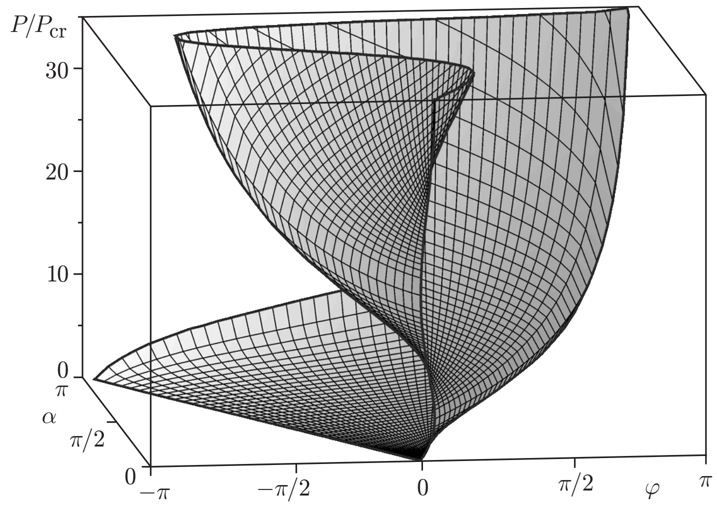

Fig. 2. Eigenvalues of the equilibrium equations for a thin bar under a follower load.

The second condition in (6) implies $\mathrm{cn}(q + F_1) = 0$, whence

$$q = (2n - 1)K(k) - F_1, \qquad n = 1, 2, 3, \dots \ . \tag{9}$$

Here $K(k)$ is a complete elliptic integral of the first kind.

Using the condition for the follower force (7), from (2) and (9) we obtain $\sin(\alpha/2) = k \,\mathrm{sn}\,(K(k), k)$ and, hence,

$$k = \sin(\alpha/2). \tag{10}$$

Relation (10) expresses obtains the modulus of the elliptic functions and integrals k in terms of the angle α. In the case of a follower force, the load at a specified angle α determines the angle φ, unlike in the case of a dead force [4], where the load determines the modulus k for a specified angle φ.

Expressions (8)–(10) imply the eigenvalue spectrum $q_n(k)$, which, in turn, determines the critical loads

$$P/P_{\mathrm{cr}} = (2/\pi)^2\{(2n - 1)K(\sin(\alpha/2)) - F[\arcsin(\sin(\varphi/2)/\sin(\alpha/2)), \sin(\alpha/2)]\}^2. \tag{11}$$

Here $P_{\mathrm{cr}} = (\pi/2)^2 EI/L^2$ is the Euler critical force and n is the mode number. Expression (11) gives the relation between the external follower force P and the slope of the force to the Ox axis φ for each mode n for a specified follower angle α. One can see from relation (11) that for each mode, variation in the magnitude of the external force leads to variation in the parameter φ within the limits

$$-\alpha \leqslant \varphi \leqslant \alpha. \tag{12}$$

For $\varphi = \pm\alpha$, we obtain the threshold values of the external force

$$P_n/P_{\mathrm{cr}} = (n - 1)^2(4/\pi)^2 K^2(\sin(\alpha/2)) \qquad (n = 1, 2), \dots, \tag{13}$$

for which transition from one mode to another occurs. The first mode ($n = 1$) begins from the zero threshold $P_1 = 0$.

For the first mode, the external force P/P_{cr} increases smoothly from 0 to the next, second threshold value P_2, and the slope of the force to the Ox axis φ varies smoothly from α to $-\alpha$. The second mode of the solution occurs when the load P exceeds the threshold value P_2, where $\varphi = -\alpha$. For the second mode ($n = 2$), the external force P/P_{cr} increases from P_2 to P_3 and the parameter φ varies inversely, from $-\alpha$ to α, etc.

Thus, the main special feature of the solution of the problem of bar bending under a follower load is a smooth transition between the modes (change of modes) with increase in the external load. One can speak of a smooth loss of stability of the bent bar. In contrast, in the problem of a bar under a dead load [4], there is no smooth transition between the modes, only one mode of the solution occurs, and higher modes can occur only under pulsed discontinuous loading. In this case, one can speak of a sudden loss of stability.

Figure 2 shows the dependence of the load $P/P_{\rm cr}$ on α and φ calculated by (11). Substituting the values of q, F_1, and k from (8)–(10) into (4) and (5), one obtains the coordinates of the bar. Generally, for specifies values of the follower angle α and the magnitude of the force P, the shape of the bar is given by the formulas

$$\begin{aligned}\frac{x}{L} &= \Big[-t + 2\,\frac{E(\operatorname{am} u) - E_1}{pK(k) - F_1}\Big]\cos\varphi + 2k\,\frac{\operatorname{cn} F_1 - \operatorname{cn} u}{pK(k) - F_1}\sin\varphi,\\ \frac{y}{L} &= 2k\,\frac{\operatorname{cn} F_1 - \operatorname{cn} u}{pK(k) - F_1}\cos\varphi - \Big[-t + 2\,\frac{E(\operatorname{am} u) - E_1}{pK(k) - F_1}\Big]\sin\varphi,\end{aligned}\tag{14}$$

where

$$F_1 = F[\arcsin(\sin(\varphi/2)/k), k], \qquad E_1 = E[\arcsin(\sin(\varphi/2)/k), k],$$

$$u = (pK(k) - F_1)t + F_1, \qquad p = 2n - 1, \qquad k = \sin(\alpha/2),$$

and the magnitude of the force is related to the parameter φ by expression (11).

Thus, each specified values of the external force P, the mode number n, and the follower angle α correspond to the bar shape determined by a single parameter, the angle φ calculated from relation (11) for the specified force P.

Coordinates of the Tip. Setting $t = 1$, from (14) we obtain the coordinates (x_1, y_1) of the unclamped end of the bar

$$\frac{x_1}{L} = \Big[2\,\frac{pE - E_1}{pK - F_1} - 1\Big]\cos\varphi + \frac{2k\operatorname{cn} F_1}{pK - F_1}\sin\varphi, \qquad \frac{y_1}{L} = \frac{2k\operatorname{cn} F_1}{pK - F_1}\cos\varphi - \Big[2\,\frac{pE - E_1}{pK - F_1} - 1\Big]\sin\varphi. \tag{15}$$

Coordinates of the Inflection Points. The points at which the second derivative d^2y/dx^2 vanishes are the inflection points of the bar axis. We denote the unknown curvilinear coordinate of this point by t_1. Using (2), we obtain the second derivative of the bar deflection line from the parametric equations (14):

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^3} = \frac{1}{\cos^3\theta}\,\frac{d\theta}{dt} = \frac{2kq\operatorname{cn} u}{\cos^3\theta}.$$

Taking into account the properties of the zeros of the Jacobi elliptic cosine $\operatorname{cn}(2m+1)K(k) = 0$ and expression (9) for the eigenvalue q, we find that

$$[(2n-1)K(k) - F_1]t_1 + F_1 = (2m+1)K(k),$$

where $n = 1, 2, 3, \ldots$ is the mode number and $m = 0, 1, 2, \ldots$ is the zero number for the elliptic cosine, which coincides with the inflection-point number. Finally, we obtain

$$t_1(\varphi) = \frac{(2m+1)K(k) - F_1}{(2n-1)K(k) - F_1}, \qquad F_1 = F[\arcsin(\sin(\varphi/2)/k), k]. \tag{16}$$

The solution mode number is related to the number of inflection points of the bar axis. For the first mode, there is one inflection point at the bar end $t_1 = 1$. For the second mode, the second inflection point appears and shifts smoothly along the bar.

Coordinates of the Compression Points. The points at which the slopes of the tangent and the compressive force to the Ox axis coincide will be called compression points. We denote the required curvilinear coordinate of this point by t_0. Setting $\theta(t_0) = -\varphi$ in the expression for the slope of the tangent (2), we obtain

$$0 = \arcsin[k\operatorname{sn}(qt_0 + F_1, k)].$$

Taking into account the properties of the zeros of the elliptic sine $\operatorname{sn} 2mK(k) = 0$ and expression (9) for the eigenvalue q, we have

$$((2n-1)K(k) - F_1)t_0 + F_1 = 2mK(k),$$

where $n = 1, 2, 3, \ldots$ is the mode number and $m = 0, 1, 2, \ldots$ is the number of the zero of the elliptic sine, which coincides with the compression-point number. Finally, we have

$$t_0(\varphi) = \frac{2mK(k) - F_1}{(2n-1)K(k) - F_1}, \qquad F_1 = F\Big[\arcsin\frac{\sin(\varphi/2)}{k}, k\Big]. \tag{17}$$

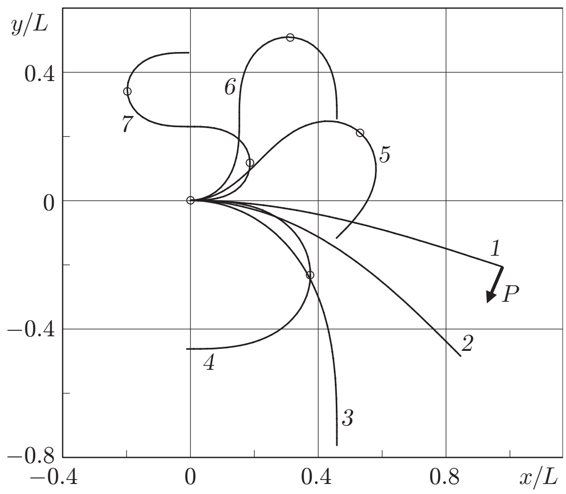

Fig. 3. Bar bending shapes for various values of the transverse follower force; first mode ($n = 1$): curve 1 refers to $\varphi = 2\pi/5$, curve 2 to $\varphi = \pi/4$, curve 3 to $\varphi = 0$, and curve 4 to $\varphi = -\pi/2$; second mode ($n = 2$): curve 5 refers to $\varphi = -\pi/4$, curve 6 to $\varphi = 0$, and curve 7 to $\varphi = \pi/2$; the compression points are denoted by circles.

Figure 3 shows bar bending shapes and compression points for various values of the applied force. For the first mode, there is one compression point at the clamped end for a slope of the force $\varphi = 0$ (for the transverse follower load, curve 3 in Fig. 3). As the force increases, the first compression point is shifted toward the unclamped end of the bar. For the second mode, the second compression point appears at the clamped end for $\varphi = 0$ (curve 6 in Fig. 3), etc.

We now consider some particular cases.

Bending of a Cantilever Bar under a Transverse Follower Load. For transverse follower loads, $\alpha = \pi/2$. From (10) it follows that

$$k = \sqrt{2}/2. \tag{18}$$

The eigenvalue spectrum (11) becomes

$$\frac{P}{P_{\rm cr}} = \Big(\frac{2}{\pi}\Big)^2\Big((2n-1)K\Big(\frac{\sqrt{2}}{2}\Big) - F\Big(\arcsin\Big(\sqrt{2}\,\sin\frac{\varphi}{2}\Big), \frac{\sqrt{2}}{2}\Big)\Big). \tag{19}$$

According to (12), the parameter φ varies in the range $-\pi/2 \leqslant \varphi \leqslant \pi/2$, determines the general curvature of the bar and it is related to the acting force P by (19). For an angle $\varphi = \pm\pi/2$, from (19) we obtain the "threshold" values of the transverse follower force

$$P_n/P_{\rm cr} = (n-1)^2((4/\pi)K(\sqrt{2}/2))^2 \approx 5.6(n-1)^2 \qquad (n = 1, 2, \ldots). \tag{20}$$

When the external force exceeds the threshold value P_n, smooth transition to the next $(n+1)$th solution mode occurs.

Expressions (14) for the coordinates of an arbitrary point of the bar x and y become

$$\begin{aligned} \frac{x}{L} &= \Big[-t + 2\,\frac{E(\operatorname{am} u) - E_1}{pK(\sqrt{2}/2) - F_1}\Big]\cos\varphi + \sqrt{2}\,\frac{\sqrt{\cos\varphi} - \operatorname{cn} u}{pK(\sqrt{2}/2) - F_1}\,\sin\varphi, \\ \frac{y}{L} &= \sqrt{2}\,\frac{\sqrt{\cos\varphi} - \operatorname{cn} u}{pK(\sqrt{2}/2) - F_1}\,\cos\varphi - \Big[-t + 2\,\frac{E(\operatorname{am} u) - E_1}{pK(\sqrt{2}/2) - F_1}\Big]\sin\varphi. \end{aligned} \tag{21}$$

Here $E(\varphi, k)$ is an incomplete elliptic integral of the second kind, $E_1 = E(\arcsin(\sin\varphi/2), \sqrt{2}/2)$, and $p = 2n-1$ $(n = 1, 2, \ldots)$. Expressions (21) were obtained using the properties of the elliptic functions $\operatorname{cn}(F_1, \sqrt{2}/2) = \sqrt{\cos\varphi}$, where $F_1 = F[\arcsin(\sqrt{2}\sin(\varphi/2)), \sqrt{2}/2]$.

Expressions (21) give a parametric description $(0 \leqslant t \leqslant 1)$ of the shape of a severely bent bar under transverse follower loading.

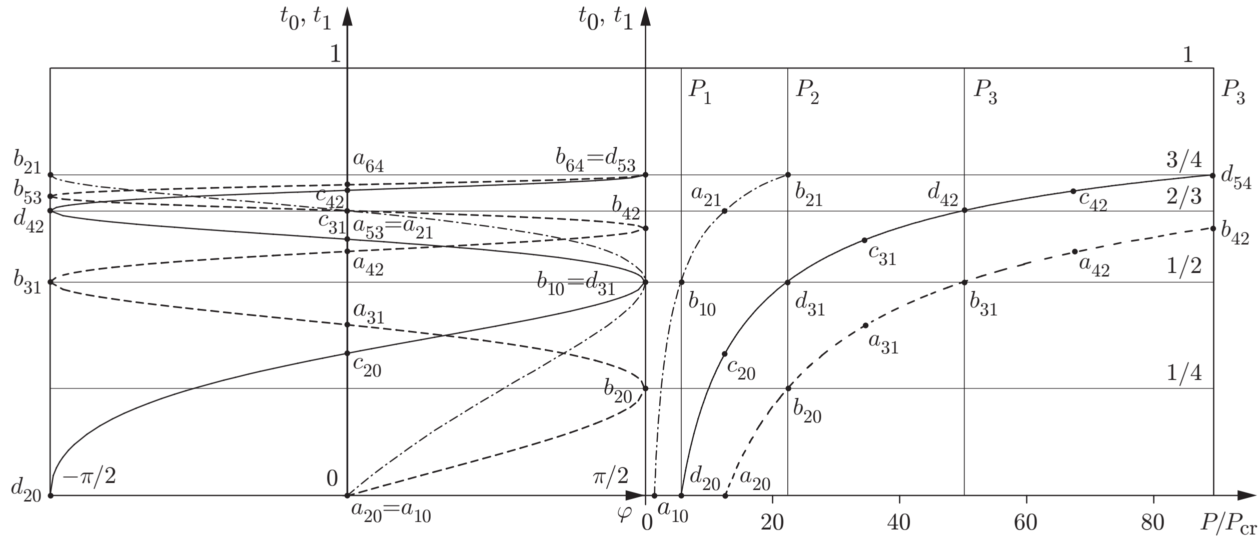

Fig. 4. Coordinates of the first two inflection points t_1 and compression points t_0 of the bar for various values of the transverse follower force: the solid curves refer to the inflection points and the dot-and-dashed and dashed curves refer to the first and second compression points, respectively.

For transverse follower forces, the coordinates t_1 and t_0 of the inflection and compression points are defined by

$$t_1(\varphi) = \frac{(2m+1)K(\sqrt{2}/2) - F_1}{(2n-1)K(\sqrt{2}/2) - F_1}, \qquad t_0(\varphi) = \frac{2mK(\sqrt{2}/2) - F_1}{(2n-1)K(\sqrt{2}/2) - F_1},$$
$$F_1 = F[\arcsin(2\sin(\varphi/2)/\sqrt{2}\,), \sqrt{2}/2]. \tag{22}$$

For the first mode ($n = 1$), there is one inflection point at the bar end $t_1 = 1$. For the second mode ($n = 2$), the second inflection point appears at the clamped end of the bar, and as the external force increases, it is shifted smoothly toward the bar end. For the third mode ($n = 3$), the third inflection point appears at the clamped end, etc.

The first compression point occurs for the first mode at the clamped end for a force parallel to the Ox axis ($\varphi = 0$) and is shifted toward the bar end. The second compression point occurs for the second mode at the clamped end for a force parallel to the Ox axis ($\varphi = 0$), etc.

Consequently, with an increase in the external follower force and with transition from one mode to another, the mode number n successively takes values of 1, 2, 3, ... for the first inflection and compression points, 2, 3, 4, ... for the second points, 3, 4, 5, ... for the third point, and so on. As the mode number changes, the integer parameter m for each inflection and compression point successively takes values of 0, 1, 2, ... and is responsible for the shift of the points along the bar.

We denote the coordinates of the compression points by a_{nm} and b_{nm} and the coordinates of the inflection points by c_{nm} and d_{nm} for $\varphi = 0$ and $\varphi = \pm\pi/2$, respectively. For $\varphi = 0$, the follower force is parallel to the Ox axis, and for $\varphi = \pm\pi/2$, it is perpendicular to this axis. It is worth noting that for $\varphi = \pm\pi/2$, expression (22) yields $F_1 = \pm K(\sqrt{2}/2)$. From (22) we obtain

$$a_{nm} \equiv t_0(0) = 2m/(2n-1), \qquad b_{nm} \equiv t_0(\pm\pi/2) = (2m+1)/(2n),$$
$$c_{nm} \equiv t_1(0) = (2m+1)/(2n-1), \qquad d_{nm} \equiv t_1(\pm\pi/2) = m/(n-1). \tag{23}$$

Figure 4 gives the coordinates of the first two inflection and compression points for the transverse follower force versus the parameter φ (on the left) and the force $P/P_{\rm cr}$ (on the right). The characteristic points at which $\varphi = 0$ and $\varphi = \pm\pi/2$ in accordance with (23) are shown. On the right, the vertical lines are drawn for the threshold values of the load $P_n/P_{\rm cr}$ according to (20). Shkutin [8] obtained similar results for bars under a transverse follower load using numerical methods.

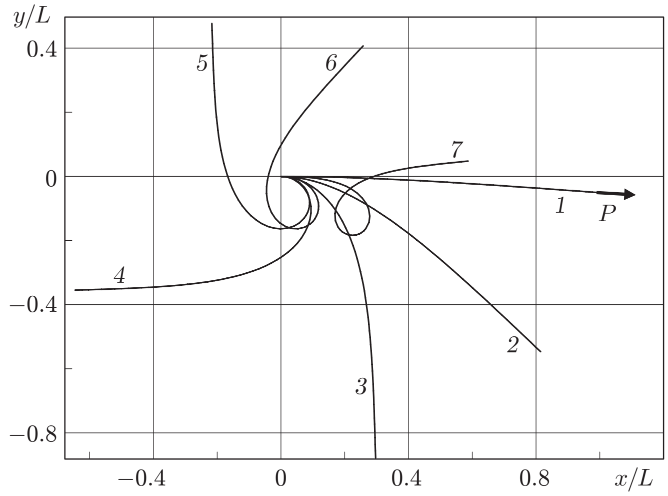

Fig. 5. Bar bending shapes for various values of the axial follower force: $\varphi = \pi - 0.07$ (1), $3\pi/4$ (2), $\pi/2$ (3), 0 (4), $-\pi/2$ (5), $-3\pi/4$ (6), and $-\pi + 0.07$ (7).

Bending of a Cantilever Bar under a Compressive Follower Load. For compressive follower loads, $\alpha = 0$. From (10), we obtain $k = 0$.

From (12), it follows that $\varphi = 0$. Passing to the limit, from (11) we obtain the critical loads

$$P_n/P_{\rm cr} = (2/\pi)^2(2n-1)^2 \tag{24}$$

equal to the well-known critical loads of a cantilever bar compressed by a dead load [4]. Since $k = 0$, the bar remains rectilinear until the load reaches the critical value. Next, when the load exceeds the Euler critical value, the bar under a dead load becomes curvilinear. In the case considered, by the definition of the axial follower force, the direction of the acting force remains constant ($\varphi = 0$) and the bar deflections cannot be found from the conditions of the problem.

Bending of a Cantilever Bar under a Tensile Follower Load. For tensile axial follower loads, $\alpha \to \pi$. In this case, from (10) we obtain $k = 1$. According to (11), the eigenvalue spectrum diverges $P_n/P_{\rm cr} \to \infty$. The bar shape remains rectilinear under any loads in this case.

For any small deviations of the angle of the follower force from π, equilibrium shapes of the bar exist. Figure 5 shows deflections of the bar for a small tensile follower angle $\alpha = \pi - 0.03$. In this case, formula (10) yields $k = 0.9999$ and the parameter varies in the range $-(\pi - 0.03) \leqslant \varphi \leqslant \pi - 0.03$. The bar bending shapes are calculated by formulas (14).

Conclusions. The solution of the geometrically nonlinear problem of a flexible bar under a follower force was obtained using the approach of [4]. The expressions for deflections (14) are written in parametric form and depend on a single parameter, the slope of the external force φ, determined by the magnitude of the force, the follower angle, and the solution mode. Unlike in [4], the solutions for deflections under dead loads depend on another parameter, the elliptic modulus k, determined by the magnitude and slope of the force and solution mode. For follower loads, transition from one mode to another occurs smoothly, in contrast to the case of dead loads.

The results of this study can be used as test cases in the development of numerical methods for solving nonlinear equations of bar bending [8].

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